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HW5

MATH 527A
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1. **The discrete heat equation**

(a) a_i and b_i are finite sequences defined for $i = 0, 1, 2, \ldots, N$. Show the *summation by parts* formula

$$\sum_{i=0}^{N-1}(a_{i+1} - a_i)b_i = a_N b_N - a_0 b_0 - \sum_{i=1}^{N} a_i(b_i - b_{i-1}).$$

Solution.

$$\begin{aligned}
\sum_{i=0}^{N-1}(a_{i+1} - a_i)b_i &= \sum_{i=0}^{N-1} a_{i+1}b_i - \sum_{i=0}^{N-1} a_i b_i \\
&= a_N b_N - a_N b_N - a_0 b_0 - \left(\sum_{i=1}^{N-1} a_i b_i - \sum_{i=0}^{N-1} a_{i+1} b_i\right) \\
&= a_N b_N - a_0 b_0 - \left(\sum_{i=1}^{N} a_i b_i - \sum_{i=0}^{N-1} a_{i+1} b_i\right) \\
\text{(reindex second sum)} &= a_N b_N - a_0 b_0 - \left(\sum_{i=1}^{N} a_i b_i - \sum_{i=1}^{N} a_i b_{i-1}\right) \\
&= a_N b_N - a_0 b_0 - \sum_{i=1}^{N} a_i(b_i - b_{i-1})
\end{aligned}$$

The discrete equation is given by

$$\begin{aligned}
\dot{u} &= \kappa\left[\frac{u_{i+1} - u_i}{l_i} + \frac{u_{i-1} - u_i}{l_{i-1}}\right] + q_i \quad i = 1, 2, \ldots, N-1 \\
u_0(t) &= u_N(t) = 0 \\
u_i(0) &= v_i
\end{aligned}$$

where $q_i(t), i = 1, 2, 3, \ldots, N-1$ is the forcing, and $l_i > 0$ for $i = 0, 1, 2, \ldots, N-1$ with $\sum_{i=0}^{N-1} l_i = L$.

(b) The "discrete" l^2 norm is defined by

$$||u||_E = \left[\sum_{i=1}^{N-1} u_i^2 l_i\right]^{1/2}.$$

Compute $\frac{d}{dt}||u||_E^2$.

Solution. First consider

$$||u||_E^2 = \sum_{i=1}^{n-1} u_i^2 l_i.$$

The derivative operator d/dt is a linear operator so it can be distributed through the sum. Therefore,

$$\frac{d}{dt}||u||_E^2 = \sum_{i=1}^{n-1} \frac{d}{dt}(u_i^2 l_i) = \sum_{i=1}^{n-1} 2\dot{u}_i u_i l_i.$$

Using the discrete heat equation for $\dot{u}_i$, we obtain

$$\frac{d}{dt}||u||_E^2 = \sum_{i=1}^{n-1} 2\kappa u_i l_i \left[\left(\frac{u_{i+1}-u_i}{l_i} + \frac{u_{i-1}-u_i}{l_{i-1}}\right) + q_i\right].$$

Using the summation by parts formula along with the boundary conditions yields

$$\frac{d}{dt}||u||_E^2 = -2\kappa \sum_{i=1}^{n-1}(u_i l_i - u_{i-1}l_{i-1})\frac{u_{i-1}-u_i}{l_{i-1}} + 2\kappa \sum_{i=1}^{n-1} u_i l_i q_i.$$

(c) If $\{q_i\} = 0$ (no forcing!) show that $||u||_E \to 0$ as $t \to \infty$.

Solution. First off, I need to establish a bound for $||u_x||_E^2 = 2\kappa \sum_{i=1}^N \left(\frac{u_i - u_{i-1}}{l_{i-1}}\right)^2 l_{i-1}$ in terms of $||u||_E^2$. The bound doesn't need to be sharp, so use dimensional analysis to get

$$\begin{aligned} ||u||_E &\leq \frac{L}{C}||u_x||_E \quad C \in \mathbb{R}^+ \\ (\implies) \ ||u||_E^2 &\leq \frac{L^2}{C^2}||u_x||_E^2 \end{aligned}$$

where C is any constant. The direction of the inequality is not a sticking point, since the bound doesn't need to be sharp. Use the result from (b)

$$\frac{d}{dt}||u||_E^2 + \frac{2\kappa C^2}{L^2}||u||_E^2 \leq \frac{d}{dt}||u||_E^2 + 2\kappa||u_x||_E^2 = 0.$$

Now, proceed with the inequality.

$$\frac{d}{dt}||u||_E^2 + \frac{2\kappa C^2}{L^2}||u||_E^2 \leq 0$$

Use the chain rule,

$$\begin{aligned} ||u||_E \frac{d}{dt}||u||_E + \frac{2\kappa C^2}{L^2}||u||_E^2 &\leq 0 \\ (\implies) \ \frac{d}{dt}||u||_E + \frac{2\kappa C^2}{L^2}||u||_E &\leq 0. \end{aligned}$$

This method works for non-zero initial energy norm $||u||_E$. Integrate the ODE and the solution is

$$||u||_E \leq ||u(0)||_E e^{-\frac{2\kappa C^2}{L^2}t} \to 0 \text{ as } t \to \infty.$$

(d) Show that the discrete heat equation from above has a unique solution.

Solution. Any two solutions u, v that satisfy the boundary and initial conditions must satisfy $||u(0) - v(0)||_E = 0$. Since the energy norm decays in time, we have

$$||u(t) - v(t)||_E = 0$$

for all t. Since E is a norm, we have $u(t) = v(t)$ for all t. Therefore the solution is unique.

(e) Assuming that $||u(0)||_E = \alpha$ and $||q(t)||_E \leq \beta$ for all t, obtain an upper bound for $||u(t)||_E$ in terms of α, β, κ, L and t.

Solution. I will proceed using Hölder's inequality.

$$\begin{aligned} \left|\sum_{i=1}^{N-1} q_i u_i l_i\right| &\leq \sum_{i=1}^{N-1} |q_i l_i u_i| \\ &\leq ||q||_E ||u||_E \\ &\leq \beta ||u||_E. \end{aligned}$$

where I have used dimensional analysis to obtain the proper factor of L.

Add the piece above to the ODE from (c) to get

$$\frac{d}{dt}||u||_E + \frac{2\kappa C^2}{L^2}||u||_E \leq \beta.$$

Solve the above equation using integrating factors to get

$$||u||_E \leq \frac{\beta L^2}{2\kappa C^2} + C_1 e^{-\frac{2\kappa C^2}{L^2}t}.$$

To find the proportionality constant C_1, use the inequality $||u(0)||_E \leq \alpha$ to get $C_1 \leq \alpha - \frac{\beta L^2}{2\kappa C^2}$. The above equation becomes

$$||u||_E \leq \frac{\beta L^2}{2\kappa C^2} + \left(\alpha - \frac{\beta L^2}{2\kappa C^2}\right) e^{-\frac{2\kappa C^2}{L^2}t}.$$

To complete the problem, I need to get rid of C.

$$
\begin{aligned}
|u_m| = |u_m - u_0| &= \left| \sum_{i=1}^{m} \left(\frac{u_i - u_{i-1}}{l_{i-1}} \right) l_{i-1} \right| \\
&\leq \sum_{i=1}^{m} \left| \left(\frac{u_i - u_{i-1}}{l_{i-1}} \right) l_{i-1} \right| \\
&\leq \sum_{i=1}^{N} \left| \left(\frac{u_i - u_{i-1}}{l_{i-1}} \right) l_{i-1} \right| \\
&\leq \left(\sum_{i=0}^{N-1} (\sqrt{l_i})^2 \right)^{1/2} \left(\sum_{i=1}^{N} \left(\frac{u_i - u_{i-1}}{l_{i-1}} \right) l_{i-1} \right)^{1/2} \\
&= \sqrt{L} ||u_x||_E.
\end{aligned}
$$

Since the argument is independent of the index, I can say the following

$$
\sup_i |u_i| = ||u||_\infty \leq \sqrt{L} ||u_x||_E.
$$

Now, I need to bound $||u||_E^2$ by $||u||_\infty$

$$
\begin{aligned}
||u||_E^2 &= \sum_{i=1}^{N-1} u_i^2 l_i \\
&\leq ||l||_1 ||u^2||_\infty \\
&\leq L ||u||_\infty^2.
\end{aligned}
$$

where I have used the fact that $(\sup_i |x_i|)^2 = \sup_i x_i^2$.

Now, I get the following bound

$$
||u||_E^2 \leq L||u||_\infty^2 \leq L^2 ||u_x||_E^2 \implies C^2 = 1 \text{ follows from (c).}
$$

The final result becomes

$$
||u||_E \leq \frac{\beta L^2}{2\kappa} + \left(\alpha - \frac{\beta L^2}{2\kappa} \right) e^{-\frac{2\kappa}{L^2} t}.
$$

2. f is a function in $C^1([0,1])$ such that $f(0) = f(1) = 0$, and it's energy norm is defined by

$$
||f||_E = \sqrt{\int_0^1 [f'(x)]^2 dx}.
$$

(a) Show that, there is a constant C such that

$$
||f||_p \leq C||f||_E
$$

for all p. Can you estimate this constant (doesn't have to be a sharp estimate)?

Solution. Since $f(0) = 0$, by the fundamental theorem of calculus, we have for all $x \in [0, 1]$

$$\begin{aligned}
|f(x)| &= |f(x) - f(0)| \\
&= \left| \int_0^x f'(t)dt \right| \\
&\leq \int_0^x |f'(t)|dt \\
&\leq \int_0^1 |f'(t)|dt \\
&\leq \left(\int_0^1 dt \right)^{1/2} \left(\int_0^1 |f'(t)|^2 dt \right)^{1/2} \\
&= 1||f||_E = ||f||_E.
\end{aligned}$$

Since the right-hand side of the above inequality is independent of x, I can say

$$||f||_\infty \leq ||f||_E.$$

I now need to use this information to control $||f||_p$. I can proceed as follows

$$\begin{aligned}
||f||_p^p &= \int_0^1 |f(t)|^p dt \\
&\leq ||1||_1 |||f|^p||_\infty \\
&= 1||f||_\infty^p = ||f||_\infty^p \\
(\implies) \quad ||f||_p &\leq ||f||_\infty.
\end{aligned}$$

The final result is

$$||f||_p \leq ||f||_\infty \leq ||f||_E$$

which implies the constant $C = 1$.

(b) If f is a C^1 function that is defined on the interval $[0, l]$ and vanishes at the endpoints. Generalize the above definition of the energy norm, and show that

$$||f||_p \leq C l^{\alpha(p)} ||f||_E$$

where C is the same constant as above and $\alpha(p)$ is an exponent that can be obtained from dimensional analysis/rescaling.

Solution. I can follow the exact same steps as above, generalizing the integral to be from 0 to l. Proceed as before

$$\begin{aligned}
|f(x)| &= |f(x) - f(0)| \\
&= \left| \int_0^x f'(t)dt \right| \\
&\leq \int_0^x |f'(t)|dt \\
&\leq \int_0^l |f'(t)|dt \\
&\leq \left(\int_0^l dt \right)^{1/2} \left(\int_0^l |f'(t)|^2 dt \right)^{1/2} \\
&= \sqrt{l} ||f||_E.
\end{aligned}$$

Since the right-hand side of the above inequality is independent of x, I can say

$$||f||_\infty \leq \sqrt{l} ||f||_E. \tag{1}$$

I now need to use this information to control $||f||_p$. I can proceed as follows

$$\begin{aligned}
||f||_p^p &= \int_0^l |f(t)|^p dt \\
&\leq ||1||_1 |||f|^p||_\infty \\
&= l||f||_\infty^p = l||f||_\infty^p \\
(\implies) \quad ||f||_p &\leq l^{1/p} ||f||_\infty.
\end{aligned}$$

The final result is

$$||f||_p \leq Cl^{1/p} ||f||_\infty \leq Cl^{1/p} \sqrt{l} ||f||_E = l^{1/p+1/2} ||f||_E.$$

which implies that $\alpha(p) = \frac{1}{p} + \frac{1}{2}$ and $C = 1$, as before.

3. A string of length L is clamped at its endpoints. It is under tension T and is subject to a transverse force per unit length $\tau(x)$. The equilibrium displacement $u(x)$ then satisfies the equation(s)

$$Tu''(x) = -\tau(x), \quad u(0) = u(L) = 0.$$

(a) Assuming that there does exist a C^2 solution for $u(x)$, find upper bounds for the maximum displacement and also the total energy $\mathrm{E}(u) = T||u||_E^2/2$ in terms of the tension T, the length L of the string, and the root-mean-squared applied forcing.

$$\bar{\tau} = \sqrt{\frac{1}{L} \int_0^L [\tau(x)]^2 dx}.$$

Also, check that your results are dimensionally consistent.

Solution. Begin by finding a bound on E(u).

$$\begin{aligned}
\mathrm{E}(u) = \frac{T}{2}||u||_E^2 &= \frac{T}{2}\int_0^L [u'(x)]^2 dx \\
&= \frac{T}{2}\left(u(x)u'(x)\Big|_0^L - \int_0^L u(x)u''(x)dx \right) \quad \text{by b.c.} \\
&= -\frac{T}{2}\int_0^L u(x)u''(x)dx \\
&= -\frac{T}{2}\int_0^L u(x)\left(-\frac{\tau(x)}{T}\right) \quad \text{by IVP definition} \\
&= \frac{1}{2}\int_0^L u(x)\tau(x)dx \\
&\le \frac{1}{2}||u||_2||\tau||_2 \quad \text{by Hölder's inequality} \\
&= \frac{1}{2}||u||_2\sqrt{L}\bar{\tau}.
\end{aligned}$$

From above, infer that

$$\begin{aligned}
\frac{T}{2}||u||_E^2 &\le \frac{1}{2}||u||_2\sqrt{L}\bar{\tau} \\
(\implies) \quad ||u||_E^2 &\le \frac{1}{T}\sqrt{L}\bar{\tau}||u||_2.
\end{aligned}$$

From problem 2b, we have

$$||u||_2 \le L^{1/2+1/2}||u||_E = L||u||_E.$$

Combine two pieces above to get

$$\begin{aligned}
||u||_E^2 &\le \frac{1}{T}\sqrt{L}\bar{\tau}L||u||_E \\
\text{divide by } ||u||_E, \quad ||u||_E &\le \frac{L^{3/2}}{T}\bar{\tau}.
\end{aligned}$$

For the maximum displacement, $||u||_\infty$, I can use problem 2b again, the step where $||u||_\infty \le ||u||_E$,

$$||u||_\infty \le ||u||_E \le \frac{L^{3/2}}{T}\bar{\tau}.$$

The energy piece is easy also, use result from above for $||u||_E^2$,

$$\mathrm{E}(u) = \frac{T}{2}||u||_E^2 \le \frac{T}{2}\frac{L^3}{T^2}\bar{\tau}^2.$$

Check dimensions on maximum displacement, $||u||_\infty$, using dimensional analysis. First, notice that $[\bar{\tau}] = \sqrt{T^2/L} = T/\sqrt{L}$.

$$[L] \le \frac{[L^{3/2}]}{[T]}\frac{[T]}{[\sqrt{L}]} = [L] \quad \square.$$

Now, for the energy, which has units $[T][L^2]$,

$$[T][L^2] \leq [T]\frac{[L^3]}{[T^2]}\frac{[T^2]}{L} = [T][L^2] \quad \square.$$

Therefore, both inequalities are dimensionally consistent.

(b) Consider the sequence of applied transverse forces

$$\tau_n(x) = \frac{n}{n^2(2x-L)^2+1}.$$

Again, assuming that we have a sequence of C^2 solutions $u_n(x)$, show that there are constants (independent of n) E_0 and u_{max} such that

$$\frac{T}{2}||u_n||_E^2 \leq \mathrm{E}_0, \quad ||u_n|| \leq u_{\max}, \quad \forall n.$$

Solution. I have no idea what τ_n is doing. Maybe there is some low-hanging fruit to be had by plotting τ_n for a few values of n.

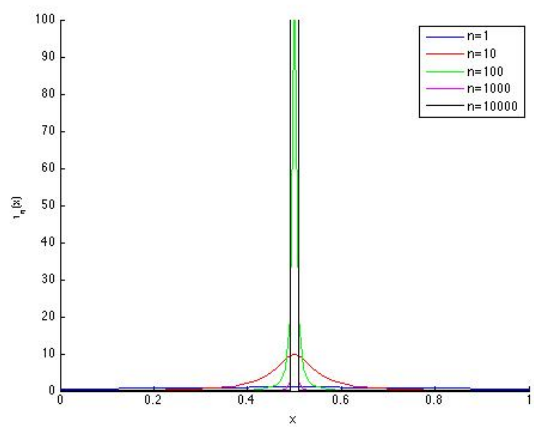

Figure 1: τ_n for several n, $L = 1$

From the picture, it becomes clear that $\lim_{n\to\infty} \tau_n(x) = C\delta(x - L/2)$. The constant C is the constant of proportionality obtained from integrating and taking the limit of the expression for τ_n.

$$\begin{aligned}
\lim_{n\to\infty}\int_0^L \frac{n}{n^2(2x-L)^2+1}dx &= \lim_{n\to\infty}\frac{1}{2}\int_{-nL}^{nL}\frac{1}{u^2+1}du \\
&= \lim_{n\to\infty}\frac{1}{2}[\arctan(nL) - \arctan(-nL)] \\
&= \frac{\pi}{2}.
\end{aligned}$$

So, $C = \frac{\pi}{2}$. So, onto the problem at hand. Start with the expression for E obtained from the first part of part (a).

$$\begin{aligned}\frac{T}{2}||u_n||_E^2 &= \frac{1}{2}\int_0^L u_n(x)\tau_n(x)dx \\ &\leq \frac{1}{2}||u_n||_\infty ||\tau_n||_1.\end{aligned}$$

Now, use what we obtained earlier (with τ_n converging to a δ-function).

$$\begin{aligned}\frac{T}{2}||u_n||_E^2 &\leq \frac{1}{2}||u_n||_\infty \left(\frac{\pi}{2}\right) \\ &\leq \frac{\pi}{4}\sqrt{L}||u_n||_E \quad \text{from problem 2b (1).}\end{aligned}$$

We know have

$$\frac{T}{2}||u_n||_E \leq \frac{\pi\sqrt{L}}{4}. \tag{2}$$

Square both sides to get

$$\begin{aligned}\frac{T^2}{4}||u_n||_E^2 &\leq \frac{\pi^2 L}{16} \\ (\Longrightarrow) \quad \frac{T}{2}||u_n||_E^2 &\leq \frac{\pi^2 L}{8T} = \mathrm{E}_0.\end{aligned}$$

For $||u_n||_\infty \leq u_{\max}$, use problem 2b (1) coupled with (2) from above.

$$\begin{aligned}\frac{T}{2}||u_n||_\infty &\leq \sqrt{L}\frac{T}{2}||u_n||_E \\ &\leq \sqrt{L}\frac{\pi\sqrt{L}}{4} \\ &= \frac{\pi L}{4} \\ (\Longrightarrow) \quad ||u_n||_\infty &\leq \frac{\pi L}{2T} = u_{\max}\end{aligned}$$

(c) What can you say about the root-mean-squared quantities $\bar{\tau}_n$? Does this contradict the conclusions from the previous two parts?

Solution. Let's look at the quantity $\bar{\tau}_n^2$.

$$\begin{aligned}\bar{\tau}_n^2 &= \frac{1}{L}\int_0^L \left(\frac{n}{n^2(2x-L)^2+1}\right)^2 dx \\ &= \frac{n}{2L}\int_{-nL}^{nL}\left(\frac{1}{u^2+1}\right)^2 du \\ &= \frac{n}{4L}\left[\frac{u}{u^2+1} + \arctan u\right]\Bigg|_{-nL}^{nL} \\ &= \frac{1}{2L}\left[\frac{n^2 L}{n^2L^2+1} + n\arctan(nL)\right]\end{aligned}$$

Clearly the limit of these quantities is unbounded as $n \to \infty$. Is it a contradiction? NO. We have obtained two bounds on the energy and maximum amplitude, and the one obtained in (b) is sharper or more restrictive. More generally although the amount of force per unit length delivered to the string is increasing with n the *amount* of total force on the string is finite ($||\tau_n||_1 \leq \pi/2$) so that fact that the energy and maximum amplitude are bounded makes sense. As a result, the interval where there is a force shrinks as $n \to \infty$ to compensate for the increase in RMS force per length.

(d) Show that the sequence of solutions $u_n(x)$ is Cauchy in the energy norm, i.e., given an $\epsilon > 0$, there is an index N such that for all $m, n > N, ||u_m - u_n||_E < \epsilon$.

Solution. Suppose that we have $T(u_n)_{xx} = -\tau_n(x)$ and $T(u_m)_{xx} = -\tau_m(x)$ and WLOG suppose that $n > m$, then the difference satisfies

$$T(u_n - u_m)_{xx} = -(\tau_n - \tau_m) = -(F_n - F_m)_x$$

with the same BC's as before. Now we multiply both sides by $u_n - u_m$ integrate by parts and find

$$T||u_n - u_m||_E^2 = \int_0^L (F_n - F_m)(u_n - u_m)_x dx$$

Using Cauchy-Schwartz on the right hand side gives

$$T||u_n - u_m||_E^2 \leq ||u_n - u_m||_E ||F_n - F_m||_2 \to 0.$$

Since F_n is Cauchy wrt L^2. Therefore u_n is Cauchy in the energy norm.

(e) What is the "physical interpretation" of the limit $n \to \infty$, and what can you conclude from the above analysis?

Solution. The physical interpretation of the limit is a delta function or *point* force that acts at only one place along the string with infinite amplitude. Our preceding analysis shows that the resultant solution is bounded and converges in the energy norm. It is not clear that it will be C^2 since this space is not complete but we can approximate it in the energy norm by successive C^2 functions. Most importantly our analysis in this case shows that the applied force need not be continuous or even bounded to have a bounded solution with finite energy norm. It appears the telling condition is that it deliver a finite amount of total force across the string which the delta function limit achieves by definition.

4. Assume that the heat equation, and the associated boundary conditions

$$\frac{1}{2}u_{xx} = u_t, \quad u(x,0) = u_0(x), u(0,t) = u(l,t) = 0$$

has a smooth solution $u(x,t)$ for all smooth initial conditions $u_0(x)$.

(a) Show that the "spatial" L^2 and energy norms, decay as a function of time, and use this to prove the uniqueness of solutions of the heat equation.

Solution. To show that the L^2 norm decays in time, consider

$$\begin{aligned}
\frac{d}{dt}||u||_2^2 &= \frac{d}{dt}\int_0^l u^2 dx \\
&= \int_0^l \frac{d}{dt}u^2 dx \quad \text{since integral is on a bounded domain} \\
&= \int_0^l 2uu_t dx \\
&= \int_0^l 2u\left(\frac{1}{2}u_{xx}\right)dx \\
&= uu_x\Big|_0^l - \int_0^l (u_x)^2 dx \\
&= -\int_0^l (u_x)^2 dx \\
(\implies) \quad ||u||_2\frac{d}{dt}||u||_2 &= -\frac{1}{2}\int_0^l (u_x)^2 dx
\end{aligned}$$

Since, $||u||_2 \geq 0$, and the rhs is ≤ 0, we can infer that $\frac{d}{dt}||u||_2 \leq 0$. So, the norm decays in time.

Now consider the energy norm,

$$\begin{aligned}
\frac{d}{dt}||u||_E^2 &= \frac{d}{dt}\int_0^l (u_x)^2 dx \\
&= \frac{d}{dt}\left(uu_x\Big|_0^l - \int_0^l uu_{xx}dx\right) \\
&= -\int_0^l \frac{d}{dt}uu_{xx}dx \\
&= -\int_0^l (u_t u_{xx} + uu_{xxt})dx \\
&= -\int_0^l u_t(2u_t)dx - \int_0^l uu_{txx}dx \\
&= -2\int_0^l (u_t)^2dx - \left(uu_{tx}\Big|_0^l\right) + \int_0^l u_x u_{tx}dx \\
&= -2\int_0^l (u_t)^2dx - 2\int_0^l u_{xx}u_t dx \\
&= -2\int_0^l (u_t)^2dx - 4\int_0^l (u_t)^2dx \\
&= -6\int_0^l (u_t)^2dx \leq 0.
\end{aligned}$$

This implies, $||u||_E$ decays in time for same reason as in the L^2 sense.

Both the energy and L^2 norms decay in time. For any two solutions u, v that solve the IVP, $||u(x,0) - v(x,0)||_E = ||u(x,0) - v(x,0)||_2 = 0$. Since the norms decay in time, also $||u(x,0) - v(x,0)||_E = ||u(x,t) - v(x,t)||_2 = 0$ for all t. Therefore, for all t, $u = v$ and the solution is unique.

(b) Show that there is a constant C such that $||u(\cdot,t)||_{L^2} \leq ||u_0||_{L^2}e^{-Ct}$. Show that for the map $S_t : u_0 \mapsto u(x,t)$ is a continuous map from $L^2([0,l])$ into itself for all $t \geq 0$.

Solution. Proceed much the same way as in problem 1. I first want to bound $||u||_2^2$ by $||u||_E^2$.

$$\begin{aligned}
|u(x,t)| &= |u(x,t) - u(0,t)| \\
&= \left|\int_0^l u_x(x,t)dx\right| \\
&\leq |u_x(x,t)|dx \\
&\leq \sqrt{l}||u_x||_2 \\
&= 2\sqrt{l}||u||_E \\
(\implies) \quad ||u||_\infty &\leq 2\sqrt{l}||u||_E \\
(\implies) \quad ||u||_\infty^2 &\leq 4l||u||_E^2.
\end{aligned}$$

Now, look at $||u||_2^2$,

$$\begin{aligned} ||u||_2^2 &= \int_0^l u^2 dx \\ &\leq ||1||_1 ||u^2||_\infty \\ &= l||u||_\infty^2. \end{aligned}$$

Put the pieces together,

$$||u||_2^2 \leq 4l||u||_\infty^2 \leq 4l^2||u||_E^2.$$

Now, look at

$$\begin{aligned} \frac{d}{dt}||u||_2^2 &= -||u||_E^2 \\ &\leq -\frac{1}{4l^2}||u||_2^2 \\ (\implies) \quad \frac{d}{dt}||u||_2 &\leq -\frac{1}{4l^2}||u||_2. \end{aligned}$$

Solve the ODE and apply initial conditions to get

$$||u||_2 \leq ||u_0||_2 e^{-\frac{1}{4l^2}t}.$$

Since $||S_t(u)||_2 = ||u||_2 \leq e^{-\frac{1}{4l^2}}||u_0||_2$, S_t is clearly bounded, therefore it is a continuous map into L^2.

5. The wave equation with dissipation is given by

$$u_{xx} = u_{tt} + bu_t$$

with the initial/boundary conditions

$$u(x,0) = u_0(x), \quad u_t(x,0) = v_0(x), \quad u(0,t) = u(l,t) = 0.$$

Show the uniqueness of smooth solutions of this equation by constructing an appropriate energy.

Solution. Assume $b > 0$. Use the following energy norm

$$||u||_E = \sqrt{\int_0^l (u_x^2 + u_t^2) dx}.$$

To show the norm decays in time,

$$\begin{aligned}
\frac{d}{dt}||u||_E^2 &= \frac{d}{dt}\int_0^l (u_x^2 + u_t^2)dx \\
&= \int_0^l \frac{d}{dt}(u_x^2 + u_t^2)dx \\
&= \int_0^l (2u_x u_{xt} + 2u_t u_{tt})dx \\
&= -2\Big(\int_0^l u_{xx}u_t - \int_0^l u_t u_{tt} dx \\
&= 2\int_0^l u_t(u_t t - u_x x)dx \\
&= -2b\int_0^l u_t^2 dx \leq 0.
\end{aligned}$$

Since $||u||_E \geq 0$, $\frac{d}{dt}||u||_E \leq 0$. So, the energy norm decays in time.

Take u, v to be solutions to the IVP. This implies $||u(x,0) - v(x,0)||_E = 0$; and since the energy norms decay in time $||u(x,t) - v(x,t)||_E = 0$ for all t. Therefore, $u = v$; i.e. the solutions are unique.